Spectral flow construction of $N = 2$ Superconformal orbifolds.

A.Belavin, V.Belavin and S.Parkhomenko

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Ten-dimensional Superstring theory unifies Standard Model and Quantum gravity. To obtain a 4-dimensional theory with Space-Time Supersymmetry (which is necessary for phenomenological reasons), we must compactify the 6 of 10 dimensions on the so-called Calabi-Yau manifold (Candelas et al).

Another equivalent approach to do the same is the compactification of 6 dimensions into $N = 2$ Superconformal Field Theory with the central charge $c = 9$, (D. Gepner).

Each of these two equivalent approaches has its own merits. Gepner's approach makes it possible to use exactly solvable N=2 SCFT models and thus obtain an explicit solution of the considered model.

The subject of the talk is a new approach to the construction of Calabi-Yau orbifolds of Fermat type required for the compactification.

The idea is to use the connection of the CY orbifolds with a class of exactly solvable models N=2 SCFT for explicitly constructing a complete set of fields in the models using the Spectral Flow Twisting (Schwimmer and Seiberg) and the requirement of Mutual Locality of the fields.

For a few examples , we construct chiral and anti-chiral rings and show that their dimensions are consistent with the dimensions of the Cohomology groups of mutually mirror CY-manifolds (including of those considered by Greene and Plesser).

•The large class of CY can be defined in manifolds $\mathbb{P}_{n_1,\dots,n_5}$ as zeroes of the quasi-homogeneous polynomials of Fermat type

$$W(x_i) := \sum_{i=1}^{5} x_i^{k_i+2}, \textit{where } W(\lambda^{n_i} x_i) = \lambda^d W(x_i) \qquad (1)$$

$$\textit{and } \sum_i n_i = d \textit{ or}, \textit{ which is the same}, \sum_{i=1}^{5} \frac{1}{k_i+2} = 1 \qquad (2)$$

•One can also obtain the new CY manifolds as orbifolds of the previous one factorizing it by an admissible group, defined as

$$G_{adm} \subset G_{tot} = \prod_i \mathbb{Z}_{k_i+2}, \ x_i \to \omega_i^{m_i} x_i, \ \textit{where } \omega_i^{k_i+2} = 1, \qquad (3)$$

$$m_i \in \mathbb{Z}, \ \sum_i \frac{m_i}{k_i+2} \in \mathbb{Z}, \ \textit{and element } (1,1,1,1,1) \in G_{adm}. \qquad (4)$$

•G_{adm} preserves the product of x_i, that is equivalent to preserving the nonvanishing holomorphic 3-form on the CY manofold.

•The geometric approach is equivalent to $N=2$ SCFT approach with $c=9$ (D. Gepner):

$$For\ Calabi-Yau\ of\ Fermat\ class\ \Rightarrow \prod_{i=1}^{5} M_{k_i}, \tag{5}$$

where M_{k_i} is Minimal model of $N=2$ SCFT with $c_i=\frac{3k_i}{k_i+2}$.

$$c=\sum_i c_i = 9 \Rightarrow\ CY\ condition\ \sum_i \frac{1}{k_i+2}=1. \tag{6}$$

•We show how to use such products of the exactly solvable models $N=2$ Minimal models to explicitly construct a set of fields in the orbifold models and thereby obtain their exact solution.

The order of the construction is as follows:

1. We take the product of Minimal models with the total central charge equal 9.

2. We choose an admissible group, we explicitly construct for it a special set of primary fields of the corresponding orbifold model. To do this we use the Spectral Flow twisting on the 1st step and the requirement of mutual locality on the 2nd.

3. Then we show that the OPE of the constructed fields is closed.

4. We also show that the mutual locality ensures the modular invariance of the partition function of the resulting orbifold.

5. Using various examples of orbifolds, we construct chiral and antichiral rings and show that they correspond to the cohomology groups of mutually mirrored CY-manifolds.

6. The last fact points to the connection of the constructed orbifold models with the corresponding σ-models on CY-manifolds.

$N = 2$ Super-Virasoro algebra.

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \\
[J_n, J_m] &= \frac{c}{3} n\delta_{n+m,0}, \\
[L_n, J_m] &= -mJ_{n+m}, \\
\{G_r^+, G_s^-\} &= L_{r+s} + \frac{r-s}{2} J_{r+s} + \frac{c}{6}(r^2 - \frac{1}{4})\delta_{r+s,0}, \\
[L_n, G_r^\pm] &= (\frac{n}{2} - r)G_{r+n}^\pm, \\
[J_n, G_r^\pm] &= \pm G_{r+n}^\pm,
\end{aligned}
\tag{7}
$$

Minimal models (M_k) of $N = 2$ SCFT.

$$c = \frac{3k}{k+2}, \quad k = 0, 1, 2, \ldots. \tag{8}$$

Primary fields in NS sector:

$$\Psi_{l,q}(z,\bar{z}) = \Phi_{l,q}(z)\bar{\Phi}_{l,q}(\bar{z}), \quad l = 0, \ldots, k, \quad q = -l, -l+2, \ldots, l,$$

$$\Delta_{l,q} = \frac{l(l+2) - q^2}{4(k+2)}, Q_{l,q} = \frac{q}{k+2}. \tag{9}$$

Primary fields in R sector:

$$\Psi^R_{l,q}(z,\bar{z}) = \Phi^R_{l,q}(z)\bar{\Phi}^R_{l,q}(\bar{z}), \quad l = 0, \ldots, k, \quad q = -l, -l+2, \ldots, l,$$

$$\Delta^R_{l,q} = \frac{l(l+2) - (q-1)^2}{4(k+2)} + \frac{1}{8}, Q^R_{l,q} = \frac{q-1}{k+2} + \frac{1}{2}. \tag{10}$$

Chiral and anti-chiral primary fields:

$$\Phi^c_l(z) \equiv \Phi_{l,l}(z), \quad \Phi^a_l(z) \equiv \Phi_{l,-l}(z) \tag{11}$$

$\mathbb{Z}_{k+2}$ group in N=2 Minimal model.

On the space of states of the model the actions of $\mathbb{Z}_{k+2}$ group are defined in holomorphic and anti-holomorphic sectors:

$$\mathbb{Z}_{k+2} = \{\hat{g}^n,\ n \in \mathbb{Z}, \hat{g} = \exp\left(\imath 2\pi J_0\right)\}, \quad or$$

$$\mathbb{Z}_{k+2} = \{\hat{\bar{g}}^n,\ n \in \mathbb{Z}, \hat{\bar{g}} = \exp\left(\imath 2\pi \bar{J}_0\right)\} \tag{12}$$

This group is cosistent to OPE of the local fields of the Minimal model.

Mutual locality of fields in $N = 2$ SCFT Minimal model.

- All diagonally composed fields from NS sector are mutually local.
- *NS* sector fields are mutually quasi-local with *R* sector fields: when one field goes around the other, there is the phase factor $(-1)^{F+\bar{F}}$, where F and $\bar{F}$ are the fermionic numbers of the *NS* field.
- *R* sector fields are mutually quasi-local: when one field goes around the other, there is the phase factor $(-1)^{F_1+\bar{F}_1+F_2+\bar{F}_2}$, where $F_{1,2}$ and $\bar{F}_{1,2}$ are the fermionic numbers of the *R* fields.
- This mutual quasi-locality structure is a particular case of a general picture valid for any $N = (1,1)$ superconformal theory.

The partition function of the Minimal model. The characters.

$$
\begin{aligned}
NS_{l,q}(\tau,z,\epsilon) &= \exp(-\imath 2\pi\epsilon)\, Tr_{\mathcal{H}^{NS}_{l,q}}(\exp(\imath 2\pi(L_0-\frac{c}{24})\tau+\imath 2\pi J_0 z)),\\
\tilde{NS}_{l,q}(\tau,z,\epsilon) &= \exp(-\imath 2\pi\epsilon)\, Tr_{\mathcal{H}^{NS}_{l,q}}((-1)^F\exp(\imath 2\pi(L_0-\frac{c}{24})\tau+\imath 2\pi J_0 z)),\\
R_{l,q}(\tau,z,\epsilon) &= \exp(-\imath 2\pi\epsilon)\, Tr_{\mathcal{H}^{R}_{l,q}}(\exp(\imath 2\pi(L_0-\frac{c}{24})\tau+\imath 2\pi J_0 z)),\\
\tilde{R}_{l,q}(\tau,z,\epsilon) &= \exp(-\imath 2\pi\epsilon)\, Tr_{\mathcal{H}^{R}_{l,q}}((-1)^F\exp(\imath 2\pi(L_0-\frac{c}{24})\tau+\imath 2\pi J_0 z)).
\end{aligned}
$$

where $(-1)^F$ is fermionic number operator.

• The characters form modular group representation (Gepner)

$$\begin{aligned}
NS_{l,q}(\tau+1,\theta,\epsilon) &= \exp\left(\imath 2\pi(\Delta_{l,t}-\frac{c}{24})\right)\tilde{NS}_{l,q}(\tau,\theta,\epsilon),\\
\tilde{NS}_{l,q}(\tau+1,\theta,\epsilon) &= \exp\left(\imath 2\pi(\Delta_{l,q}-\frac{c}{24})\right)NS_{l,q}(\tau,\theta,\epsilon),\\
R_{l,q}(\tau+1,\theta,\epsilon) &= \exp\left(\imath 2\pi(\Delta^R_{l,q}-\frac{c}{24})\right)R_{l,q}(\tau,\theta,\epsilon),\\
\tilde{R}_{l,q}(\tau+1,\theta,\epsilon) &= \exp\left(\imath 2\pi(\Delta^R_{l,q}-\frac{c}{24})\right)\tilde{R}_{l,q}(\tau,\theta,\epsilon),\\
NS_{l,q}(-\frac{1}{\tau},\frac{\theta}{\tau},\epsilon+\frac{c\theta^2}{6\tau}) &= \sum_{l',q'} S_{l,q}^{l',q'}NS_{l',q'}(\tau,\theta,\epsilon),\\
\tilde{NS}_{l,q}(-\frac{1}{\tau},\frac{\theta}{\tau},\epsilon+\frac{c\theta^2}{6\tau}) &= \sum_{l',q'} S_{l,q}^{l',q'-1}R_{l',q'}(\tau,\theta,\epsilon),\\
R_{l,q}(-\frac{1}{\tau},\frac{\theta}{\tau},\epsilon+\frac{c\theta^2}{6\tau}) &= \sum_{l',q'} S_{l,q-1}^{l',q'}\tilde{NS}_{l'q'}(\tau,\theta,\epsilon),\\
\tilde{R}_{l,q}(-\frac{1}{\tau},\frac{\theta}{\tau},\epsilon+\frac{cz^2}{6\tau}) &= -\imath\sum_{l',q'} S_{l,q-1}^{l',q'-1}\tilde{R}_{l',q'}(\tau,\theta,\epsilon), \qquad (14)
\end{aligned}$$

$$S_{l,t}^{l',q'} = \sin(\pi\frac{(l+1)(l'+1)}{k+2})\exp(\imath\pi\frac{qq'}{k+2}). \qquad (15)$$

•The partition function of the Minimal model:

$$Z(\tau,\bar{\tau},0,0) = \sum_{l,q}(NS_{l,q}(\tau,0)NS^*_{l,q}(\bar{\tau},0) + \tilde{NS}_{l,q}(\tau,0)\tilde{NS}^*_{l,q}(\bar{\tau},0) +$$
$$R_{l,q}(\tau,0)R^*_{l,q}(\bar{\tau},0) + \tilde{R}_{l,q}(\tau,0)\tilde{R}^*_{l,q}(\bar{\tau},0)) \qquad (16)$$

is modular invariant.

•It gets contributions from the subspace of mutually local fields:

$$\mathcal{H}_{loc} = \mathcal{H}_0^{NS} \oplus \mathcal{H}_2^{NS} \oplus \mathcal{H}_1^{R} \oplus \mathcal{H}_3^{R} \qquad (17)$$

where $\mathcal{H}_0^{NS}$ is the subspace of $\mathcal{H}^{NS}$ with even fermion numbers F and $\bar{F}$, $\mathcal{H}_2^{NS}$ is the subspace of $\mathcal{H}^{NS}$ with odd fermion numbers F and $\bar{F}$, the spaces $\mathcal{H}_{1,3}^{R}$ have similar meaning for R sector fields.

Conformal Bootstrap and $N = (2,2)$ Minimal models.

$N = (2,2)$ **Minimal model satisfies all axioms of conformal bootstrap, that is :**

• The model has a complete set of local fields that form a closed OPE algebra that satisfies the associativity condition;

• The fields in this set are mutually local;

•This set of fields ensures the modular invariance of the partition function of the model.

Spectral flow construction of primary fields in M_k.

$N=2$ **Super-Virasoro algebra possesses Spectral flow automorphism.**

The transformation

$$\begin{aligned} \tilde{G}^{\pm}_r &= U^{-t} G^{\pm}_r U^t = G^{\pm}_{r \pm t}, \\ \tilde{J}_n &= U^{-t} J_n U^t = J_n + \frac{c}{3} t \delta_{n,0}, \\ \tilde{L}_n &= U^{-t} L_n U^t = L_n + t J_n + \frac{c}{6} t^2 \delta_{n,0}, \end{aligned} \tag{18}$$

keeps the above commutation relations unchanged.

•In what follows, we will assume that $t \in \frac{1}{2}\mathbb{Z}$.

Extremal vectors and anti-chiral primary fields.

$$E_l^-(z) = G^-_{\frac{1}{2}-l} \dots G^-_{-\frac{1}{2}} \Phi_l^c(z),$$
$$G_r^+ E_l^- = 0, r \geq \frac{1}{2} + l, \ G_r^- E_l^- = 0, r \geq -\frac{1}{2} - l,$$
$$J_n E_l^- = L_n E_l^- = 0, n \geq 1. \qquad (19)$$

•The vector

$$G^-_{-\frac{1}{2}-l} E_l^-(z) = 0 \qquad (20)$$

since it is a singular vector for the representation generated by the chiral primary field Φ_l^c.
•Using this we can construct the anti-chiral primary field $\Phi_l^a(z)$ applying the spectral flow operator as follows

$$\Phi_l^a(z) = U^l E_l^-(z), \ \textit{and check that } G^-_{-\frac{1}{2}} \Phi_l^a(z) = 0. \qquad (21)$$

Extremal vector and Borel subalgebra.

• The extremal vector E_l^- is connected with one of possible Borel subalgebra in $N=2$ Super-Virasoro algebra.
This subalgebra is generated by

$$\begin{aligned} G_r^+, r \geq \frac{1}{2}+l,\ G_r^-, r > -\frac{1}{2}-l, \\ J_n,\ L_n, n \geq 0. \end{aligned} \tag{22}$$

All primary fields from the chiral primary field

- The construction of anti-chiral primary can be generalized as:

$$\Phi_{l,q}(z) = U^t G^-_{\frac{1}{2}-t} \dots G^-_{-\frac{1}{2}} \Phi^c_l(z) = (UG^-_{-\frac{1}{2}})^t \Phi^c_l(z).$$

$$q = l - 2t \tag{23}$$

This gives realization of any primary field $\Phi_{l,q}(z)$, if $0 \leq t \leq l$, and

$$\Phi_{\tilde{l},\tilde{q}}(z) = U^t G^-_{\frac{1}{2}-t} \dots G^-_{-\frac{1}{2}} \Phi^c_l(z) = (UG^-_{-\frac{1}{2}})^{t-l-1} U (UG^-_{-\frac{1}{2}})^l \Phi^c_l(z) (24)$$

(where $\tilde{l} = k - l, \tilde{q} = k + 2 + l - 2t$ and $G^-_{\frac{1}{2}-l-1}$ is missed) gives realization of the primary field $\Phi_{\tilde{l},\tilde{q}}(z)$, if $l + 1 \leq t \leq k + 1$.

- R sector primary fields are generated from NS sector primary fields:

$$\Psi^R_{l,q}(z, \bar{z}) = U^{\frac{1}{2}} \bar{U}^{\frac{1}{2}} \Psi_{l,q}(z, \bar{z}). \tag{24}$$

Spectral flow construction and mutual locality.

The above construction of all primary fields, which uses a spectral flow twisting and the bosonisation of $U(1)$ current,

turns out to be very effective not only for constructing all primary fields from the chiral primaries,

But also for calculating the violation of their mutual locality.

Product models and dioganal $N = 2$ algebra

$$M_{\vec{k}} = \prod_{i=1} M_{k_i}, \;\; c_{tot} = \sum_i c_i = 9,$$
$$L_{tot,n} = \sum_i L_{(i),n}, \;\; J_{tot,n} = \sum_i J_{(i),n}, \;\; G^{\pm}_{tot,r} = \sum_i G^{\pm}_{(i),r}. \tag{25}$$

•The representation space of this algebra is defined only as the product of either NS- or R-representations of minimal models M_{k_i}. The same is true for the anti-holomorphic sector.

•NS and R primary fields in the product model $M_{\vec{k}}$ are given by:

$$\Psi^{NS}_{\vec{l},\vec{q}}(z,\bar{z}) = \prod_i \Psi^{NS}_{l_i,q_i}(z,\bar{z}),$$
$$\Psi^{R}_{\vec{l},\vec{q}}(z,\bar{z}) = \prod_i \Psi^{R}_{l_i,q_i}(z,\bar{z}) = \prod_i U_i^{\frac{1}{2}} \bar{U}_i^{\frac{1}{2}} \Psi^{NS}_{l_i,q_i}(z,\bar{z}). \tag{26}$$

•Only products of fields of this type are consistent with the action of the diagonal $N = 2$ Virasoro superalgebra.

•The descendant fields are generated from the primary fields above by the creation generators of the $N = 2$ virasoro superalgebras of the M_{k_i} models.

•The NS sector fields from (26) are mutually local,
while R sector fields from (26) are mutually quasi-local with themselves and with NS sector fields.

•The OPE is associative and closed.

•The subspace of mutually local fields in the product models can be determined similary to the Minimal model: the fields are mutually local if their fermion numbers F and $\bar{F}$ are equal.

•Similar to the Minimal model, this constraint allows one to construct a partition function which is modular invariant.

Admissible group for an orbifold model.

To construct a new $N = 2$ SCFT starting from the $M_{\vec{k}}$ product we indroduce the so-called admissible group G_{adm}.

•The discrete symmetry group of $M_{\vec{k}}$ is defined as follows

$$G_{tot} = \prod_i \mathbb{Z}_{k_i+2} = \{\prod_i \hat{g}_i^{n_i}, \; n_i \in \mathbb{Z}, \hat{g}_i = \exp\left(\imath 2\pi J_{(i),0}\right)\} \qquad (27)$$

•Admissible group:

$$G_{adm} = \{\vec{w} = \sum_{a=0}^{M-1} w^a \vec{\beta}_a, \; w^a \in \mathbb{Z}, \; \sum_i \frac{\beta_{ai}}{k_i + 2} \in \mathbb{Z}, \; \beta_{ai} \in \mathbb{Z},$$
$$\vec{\beta}_0 = (1, 1, 1, ..., 1), \; \sum_i \frac{1}{k_i + 2} = 1\} \subset G_{tot}. \quad (28)$$

•The group G_{adm} coincides with the group that preserves the nowhere vanishing holomorphic $(3, 0)$-form of the corresponding $N = 2$ SCFT, defined as a σ -model on CY orbifold in $\mathbb{P}_{n_1,...,n_5}$.

Construction of primary fields of orbifold model.

•**At the first step,** we expand the state space of the $M_{\vec{k}}$ product model by adding twisted fields of the form

$$\Psi^{NS}_{\vec{l},\vec{t},\vec{w}}(z,\bar{z}) = V_{\vec{l},\vec{t}+\vec{w}}(z)\bar{V}_{\vec{l},\vec{t}}(\bar{z}), \tag{29}$$

$$\bar{V}_{\vec{l},\vec{t}}(\bar{z}) = \prod_i \bar{V}_{l_i,t_i}(\bar{z}), \;\; \bar{V}_{l_i,t_i}(z) = (UG^-_{-\frac{1}{2}})_i^{t_i}\bar{\Phi}^c_{l_i}(z), \;\; 0 \leq t_i \leq l_i, \tag{30}$$

$$V_{\vec{l},\vec{t}+\vec{w}}(z) = \prod_{i=1}^{5} V_{l_i,t_i+w_i}(z),$$
$$V_{l_i,t_i+w_i}(z) = (UG^-_{-\frac{1}{2}})_i^{t_i+w_i}\Phi^c_{l_i}(z), \;\; \mathit{if}\;\; 0 \leq t_i + w_i \leq l_i,$$
$$\mathit{or}$$
$$V_{l_i,t_i+w_i}(z) \equiv (UG^-_{-\frac{1}{2}})_i^{t_i+w_i-l_i-1}U_i(UG^-_{-\frac{1}{2}})_i^{l_i}\Phi^c_{l_i}(z),$$
$$\mathit{if}\;\; l_i + 1 \leq t_i + w_i \leq k_i + 1. \tag{31}$$

- **At the second step** we require mutual locality of the fields

$$Q^{\vec{\beta}_a}_{\vec{l},\vec{t}} - \sum_i \frac{\beta_{ai} w_i}{k_i + 2} \in \mathbb{Z}, \;\; a = 0, ..., M-1,$$

$$\textit{where } Q^{\vec{\beta}_a}_{\vec{l},\vec{t}} = \sum_i \frac{\beta_{ai} q_i}{k_i + 2}, \;\; q_i = l_i - 2t_i. \tag{32}$$

- **At the third step** we generate R sector fields:

$$\Psi^R_{\vec{l},\vec{t},\vec{w}}(z, \bar{z}) = \prod_i U_i^{\frac{1}{2}} \bar{U}_i^{\frac{1}{2}} \Psi^{NS}_{\vec{l},\vec{t},\vec{w}}(z, \bar{z}). \tag{33}$$

• Using the explicit expressions (29), (33) and (32) one can show that mutual locality equations are

$$Q_{\vec{l},\vec{t}}^{\vec{\beta}_a} - \sum_i \frac{\beta_{ai} w_i}{k_i + 2} \in \mathbb{Z}, \ \ a = 0, ..., M-1,$$
$$F + \bar{F} + \sum_i w_i \in 2\mathbb{Z}. \qquad (34)$$

• These equations single out the set of the mutually local fields for the orbifold model.
• OPE of the constructed model is closed.

• Below we use the spectral flow construction above to find (c, c) states with charges $(1, 1)$ and (a, c) states with charges $(-1, 1)$ in several examples of orbifolds.

• We show their connection with the cohomology groups of the mirror pairs of CY manifolds.

Orbifold partition function and mutual locality.

- From the obtained set of mutually local fields , we construct the partition function of the orbifold model as follows

$$Z_{G_{adm}} = \frac{1}{|G_{adm}|} \sum_{\vec{w}\in G_{adm}} \sum_{\vec{l},\vec{t}} \prod_{a=0}^{M-1} \delta(Q^{\vec{\beta}_a}_{\vec{l},\vec{t}} - \sum_i \frac{\beta_{ai} w_i}{k_i+2})$$
$$(NS_{\vec{l},\vec{t}+\vec{w}} NS^*_{\vec{l},\vec{t}} + R_{\vec{l},\vec{t}+\vec{w}} R^*_{\vec{l},\vec{t}} +$$
$$\exp[\imath\pi(\sum_i w_i)](\tilde{NS}_{\vec{l},\vec{t}+\vec{w}} \tilde{NS}^*_{\vec{l},\vec{t}} + \tilde{R}_{\vec{l},\vec{t}+\vec{w}} \tilde{R}^*_{\vec{l},\vec{t}})), \qquad (35)$$

where

$$NS_{\vec{l},\vec{q}} \equiv \prod_i NS_{l_i,q_i}, \quad \tilde{NS}_{\vec{l},\vec{q}} \equiv \prod_i \tilde{NS}_{l_i,q_i},$$
$$R_{\vec{l},\vec{q}} \equiv \prod_i R_{l_i,q_i}, \quad \tilde{R}_{\vec{l},\vec{q}} \equiv \prod_i \tilde{R}_{l_i,q_i}. \qquad (36)$$

- By direct verification, we have seen that this is a modular invariant.

This completes the construction of the $N=2$ SUSY orbifold model.

$H^{1,2}$ for two generators admissible group.

$$G_{adm} = \{n\vec{\alpha} + m\vec{\beta},\ \vec{\alpha} = (1,1,1,1,1),\ \vec{\beta} = (\beta_1, ..., \beta_5).\} \qquad (37)$$

• To construct $H^{1,2}$ group one neeeds to find (c,c) fields with $Q = \bar{Q} = 1$.

• Chiral primary in anti-holomorphic sector must obeys:

$$q_i = l_i,\ \sum_i \frac{l_i}{k_i+2} = 1,\ \ l_i = 0, ..., k_i. \qquad (38)$$

• Chiral primary in holomorphic sector must obeys:

$$\sum_i \frac{\tilde{l}_i}{k_i+2} = 1,\ \ where\ \ \tilde{l}_i = l_i\ \ if\ \ n + m\beta_i = 0\ \ mod\ \ k_i + 2,\ \ or$$
$$\tilde{l}_i = k_i - l_i\ \ if\ \ l_i + 1 = n + m\beta_i\ \ and\ \ n + m\beta_i \neq 0\ \ mod\ \ k_i + 2. \ (39)$$

• The mutual locality equations:

$$\sum_i \frac{l_i}{k_i+2} \in \mathbb{Z},\ \sum_i \beta_i \frac{l_i - m\beta_i}{k_i+2} \in \mathbb{Z}. \qquad (40)$$

Algorithm of finding $H^{1,2}$ states.

1. Find all the vectors $\vec{l} = (l_1, ..., l_5)$ satisfying the equations:

$$\sum_i \frac{l_i}{k_i+2} = 1, \ \sum_i \beta_i \frac{l_i - m\beta_i}{k_i+2} \in \mathbb{Z} \tag{41}$$

2. Among the found vectors $\vec{l}$ select those for which there are vectors $(\tilde{l}_1, ..., \tilde{l}_5)$ constructed according to the rules:

$$\sum_i \frac{\tilde{l}_i}{k_i+2} = 1, \ and$$
$$\tilde{l}_i = l_i \ if \ n + m\beta_i = 0 \ mod \ k_i + 2,$$
$$or$$
$$\tilde{l}_i = k_i - l_i \ if \ l_i + 1 = n + m\beta_i \ and \ n + m\beta_i \neq 0 \ mod \ k_i + 2. \tag{42}$$

3. Write out the pairs $(\vec{l}_L = (\tilde{l}_1, ..., \tilde{l}_5), \vec{l}_R = \vec{l})$.

The list of $H^{1,2}$ states in the particular case $M_{\vec{k}} = M_{3,3,3,3,3}$, $\vec{\alpha} = (1,1,1,1,1)$, $\vec{\beta} = (0,0,0,1,4)$.

• **Untwisted sector- 25 states:**

$$(\vec{l}_L, \vec{l}_R) = \\ \{((1,1,1,1,1),(1,1,1,1,1)),((3,2,0,0,0),(3,2,0,0,0)), \\ ((3,1,1,0,0),(3,1,1,0,0)),((3,0,0,1,1),(3,0,0,1,1)), \\ ((2,2,1,0,0),(2,2,1,0,0)),((1,0,0,2,2),(1,0,0,2,2)), \\ ((2,1,0,1,1),(2,1,0,1,1)) \textit{ and permut.}\} \quad (43)$$

• **Twisted sector** $(n = 0, m = 1)$ - **6 states:**

$(\vec{l}_L, \vec{l}_R) = \{((1,1,0,3,0),(1,1,0,0,3)),((2,0,0,3,0),(2,0,0,0,3))$ *and* ...

• **Twisted sector** $(n = 0, m = 2)$ - **6 states:**

$(\vec{l}_L, \vec{l}_R) = \{((2,0,0,2,1),(2,0,0,1,2)),((1,1,0,2,1),(1,1,0,1,2))$ *and* ...

• **Twisted sector** $(n = 0, m = 3)$ - **6 fields:**

$(\vec{l}_L, \vec{l}_R) = \{((2,0,0,1,2),(2,0,0,2,1)),((1,1,0,1,2),(1,1,0,2,1))$ *and* ...

• **Twisted sector** $(n = 0, m = 4)$ - **6 states:**

$(\vec{l}_L, \vec{l}_R) = \{((2,0,0,0,3),(2,0,0,3,0)),((1,1,0,0,3),(1,1,0,3,0))$ *and* ...

The total number of the chiral primary fields of $Q = \bar{Q} = 1$ equals $25 + 24 = 49$.

Monomials generating polynomial part to $H^{1,2}$.

$$e_1 = x_1^3x_2^2,\ e_2 = x_1^3x_3^2,\ e_3 = x_2^3x_3^2,\ e_4 = x_1^2x_2^3,\ e_5 = x_1^2x_3^3,\ e_6 = x_2^2x_3^3,$$
$$e_7 = x_1^3x_2x_3,\ e_8 = x_1x_2^3x_3,\ e_9 = x_1x_2x_3^3,\ e_{10} = x_1^2x_2^2x_3,\ e_{11} = x_1^2x_2x_3^3,$$
$$e_{12} = x_1x_2^2x_3^2,\ e_{13} = x_1^3x_4x_5,\ e_{14} = x_2^3x_4x_5,\ e_{15} = x_3^3x_4x_5,$$
$$e_{16} = x_1^2x_2x_4x_5,\ e_{17} = x_1^2x_3x_4x_5,\ e_{18} = x_2^2x_3x_4x_5,\ e_{19} = x_1x_2^2x_4x_5,$$
$$e_{20} = x_1x_3^2x_4x_5,\ e_{21} = x_2x_3^2x_4x_5,\ e_{22} = x_1x_4^2x_5^2,\ e_{23} = x_2x_4^2x_5^2,$$
$$e_{24} = x_3x_4^2x_5^2,\ e_{25} = x_1x_2x_3x_4x_5. \quad (48)$$

Laurent monomials generating nonpolynomial part of $H^{1,2}$.

$$
\begin{gathered}
g_1 = x_1^{-1} x_2 x_3^3 x_4 x_5,\ g_2 = x_1 x_2^{-1} x_3^3 x_4 x_5,\ g_3 = x_1^3 x_2 x_3^{-1} x_4 x_5,\\
g_4 = x_1^{-1} x_2^2 x_3^2 x_4 x_5,\ g_5 = x_1^2 x_2^{-1} x_3^2 x_4 x_5,\ g_6 = x_1^2 x_2^2 x_3^{-1} x_4 x_5,\\
g_7 = x_1^{-1} x_2^3 x_3 x_4 x_5,\ g_8 = x_1 x_2^3 x_3^{-1} x_4 x_5,\ g_9 = x_1^3 x_2^{-1} x_3 x_4 x_5,\\
g_{10} = x_1^{-1} x_3^2 x_4^2 x_5^2,\ g_{11} = x_2^{-1} x_3^2 x_4^2 x_5^2,\ g_{12} = x_1^2 x_3^{-1} x_4^2 x_5^2,\\
g_{13} = x_1^{-1} x_2 x_3 x_4^2 x_5^2,\ g_{14} = x_1 x_2^{-1} x_3 x_4^2 x_5^2,\ g_{15} = x_1 x_2 x_3^{-1} x_4^2 x_5^2,\\
g_{16} = x_1^{-1} x_2^2 x_4^2 x_5^2,\ g_{17} = x_2^2 x_3^{-1} x_4^2 x_5^2,\ g_{18} = x_1^2 x_2^{-1} x_4^2 x_5^2,\\
g_{19} = x_1^{-1} x_4^3 x_5^3,\ g_{20} = x_2^{-1} x_4^3 x_5^3,\ g_{21} = x_3^{-1} x_4^3 x_5^3,\ g_{22} = x_1^{-1} x_2^3 x_3^3,\\
g_{23} = x_1^3 x_2^{-1} x_3^3,\ g_{24} = x_1^3 x_2^3 x_3^{-1}. \qquad (49)
\end{gathered}
$$

$H^{1,1}$ group finding.

- **To construct $H^{1,1}$ group one needs to find (a,c) fields with $Q=-1,\ \bar{Q}=1$.**
- Chiral primary in anti-holomorphic sector must obeys:

$$q_i = l_i,\ \sum_i \frac{l_i}{k_i+2} = 1,\ l_i = 0, ..., k_i. \tag{50}$$

- Anti-chiral primary in holomorphic sector must obeys:

$$-\sum_i \frac{\tilde{l}_i}{k_i+2} = -1,\ where\ \tilde{l}_i = l_i\ if\ l_i = n + m\beta_i,\ or$$
$$\tilde{l}_i = k_i - l_i\ if\ n + m\beta_i = k_i + 1. \tag{51}$$

- The mutual locality equations:

$$\sum_i \frac{l_i}{k_i+2} \in \mathbb{Z},\ \sum_i \beta_i \frac{l_i - m\beta_i}{k_i+2} \in \mathbb{Z}. \tag{52}$$

Algorithm of finding $H^{1,1}$ states.

1. Find all the vectors $\vec{l} = (l_1, ..., l_5)$ satisfying the equations:

$$\sum_i \frac{l_i}{k_i+2} = 1, \ \sum_i \beta_i \frac{l_i - m\beta_i}{k_i+2} \in \mathbb{Z} \tag{53}$$

2. Among the found vectors $\vec{l}$ select those for which there are vectors $(\tilde{l}_1, ..., \tilde{l}_5)$ constructed according to the rules:

$$\begin{aligned} \sum_i \frac{\tilde{l}_i}{k_i+2} &= 1, \ \textit{and} \\ \tilde{l}_i = l_i \ \textit{if} \ n + m\beta_i &= l_i, \\ &\textit{or} \\ \tilde{l}_i = k_i - l_i \ \textit{if} \ n + m\beta_i &= k_i + 1. \end{aligned} \tag{54}$$

3. Write out the pairs $(\vec{l}_L = (\tilde{l}_1, ..., \tilde{l}_5), \vec{l}_R = \vec{l})$.

The list of $H^{1,1}$ states in the particular case

$M_{\vec{k}} = M_{3,3,3,3,3}$, $\vec{\alpha} = (1,1,1,1,1)$, $\vec{\beta} = (0,0,0,1,4)$.

- There are no states in untwisted sector.
- twisted sector ($n = 1, m = 0$)-there is 1 field:

$$(\vec{l}_L, \vec{l}_R) = ((1,1,1,1,1),(1,1,1,1,1)), \tag{55}$$

- twisted sector ($n = 0, m = 2$)-there is 1 field:

$$(\vec{l}_L, \vec{l}_R) = ((0,0,0,2,3),(0,0,0,2,3)), \tag{56}$$

- twisted sector ($n = 0, m = 3$)-there is 1 field:

$$(\vec{l}_L, \vec{l}_R) = ((0,0,0,3,2),(0,0,0,3,2)), \tag{57}$$

- twisted sector ($n = 1, m = 1$)-there is 1 field:

$$(\vec{l}_L, \vec{l}_R) = ((1,1,1,2,0),(1,1,1,2,0)), \tag{58}$$

- twisted sector ($n = 1, m = 4$)-there is 1 field:

$$(\vec{l}_L, \vec{l}_R) = ((1,1,1,0,2),(1,1,1,0,2)). \tag{59}$$

$$\textit{THANK YOU FOR YOUR ATTENTION !} \tag{60}$$